8.333 Fall 2025 Recitations 4-6a: Kinetic theory recap

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These notes are largely a conglomeration of the previous years' recitation notes by Julien Tailleur, Amer Al-Hiyasat, and Sara Dal Cengio.

References. All the essential information in these recitations can be found in Chapter 3 of Mehran Kardar's *Statistical Physics of Particles*. Also see lectures 7-11 of his 8.333 OCW notes, and lectures 4-11 of Julien Tailleur's notes.

Contents

1 Intro, definitions and notations

We are concerned with extremely the high-dimensional problem of many-particle ($N \gtrsim 10^{23}$) Hamiltonian dynamics. How do we reduce the complicated microscopic dynamics to the simpler evolution of macroscopic quantities?

Work with N particles in 3 dimensions. Suppose particle i has position $\mathbf{q}_i = (q_1^x, q_1^y, q_1^z)$ and momentum $\mathbf{p}_i = (p_1^x, p_1^y, p_1^z)$. (On the blackboard, I use replace the boldfaced letter with the arrow version, i.e. $\mathbf{q}_i \to \vec{q}_i$.) Use the notation

$$\mathbf{Q} \equiv (\mathbf{q}_1, \mathbf{q}_2, \dots, \mathbf{q}_N)\,, \qquad \mathbf{P} \equiv (\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N)\,, \qquad \mathbf{\Gamma} \equiv (\mathbf{q}_1, \dots, \mathbf{q}_N, \mathbf{p}_1, \dots, \mathbf{p}_N)\,. \tag{1}$$

Also make the definitions

$$\mathbf{q}_{ij} \equiv \mathbf{q}_i - \mathbf{q}_j\,, \qquad q_{ij} \equiv |\mathbf{q}_{ij}|\,, \qquad p_i \equiv |\mathbf{p}_i|\,, \qquad d\Gamma_i \equiv d^3q_i d^3p_i\,. \tag{2}$$

The particles evolve under Hamiltonian dynamics with the Hamiltonian

$$H(\mathbf{Q},\mathbf{P}) = \sum_{i=1}^{N}\left[\frac{p_i^2}{2m} + U(\mathbf{q}_i) + \frac{1}{2}\sum_{j\neq i} V(q_{ij})\right] \equiv H_1(\mathbf{Q},\mathbf{P}) + \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i} V(q_{ij})\,. \tag{3}$$

In particular, we consider a two-body interaction potential $V(q)$ which is spherically symmetric, i.e. only depending on q rather than $\mathbf{q}$. Also define

$$U_i \equiv U(\mathbf{q}_i)\,, \qquad V_{ij} \equiv V(q_{ij})\,. \tag{4}$$

We will use the Poisson bracket, which for operators $A(\mathbf{Q},\mathbf{P})$ and $B(\mathbf{Q},\mathbf{P})$ is defined as

$$\{A,B\} \equiv \sum_{i=1}^{N}\left[\frac{\partial A}{\partial \mathbf{q}_i}\cdot\frac{\partial B}{\partial \mathbf{p}_i}-\frac{\partial A}{\partial \mathbf{p}_i}\cdot\frac{\partial B}{\partial \mathbf{q}_i}\right] = \frac{\partial A}{\partial \mathbf{Q}}\cdot\frac{\partial B}{\partial \mathbf{P}}-\frac{\partial A}{\partial \mathbf{P}}\cdot\frac{\partial B}{\partial \mathbf{Q}}\,. \tag{5}$$

It has the following properties (for operators A, B, C and scalar λ), which we will use:

$$\{B,A\} = -\{A,B\} \qquad \text{(antisymmetry)} \tag{6}$$
$$\{A,B+\lambda C\} = \{A,B\}+\lambda\{A,C\} \qquad \text{(bilinearity)} \tag{7}$$
$$\{A+\lambda C,B\} = \{A,B\}+\lambda\{C,B\} \qquad \text{(bilinearity)}\,. \tag{8}$$

The probability density over phase space $\rho(\mathbf{Q},\mathbf{P};t)$ is the probability density of particles at phase space point $(\mathbf{Q},\mathbf{P})$ at time t. It evolves according to the Liouville equation, whose derivation proceeds as follows:

$$0 = \frac{d\rho}{dt} \qquad \text{(Liouville theorem)} \tag{9}$$
$$= \frac{\partial\rho}{\partial t}+\dot{\mathbf{Q}}\cdot\frac{\partial\rho}{\partial\mathbf{Q}}+\dot{\mathbf{P}}\cdot\frac{\partial\rho}{\partial\mathbf{P}} = \frac{\partial\rho}{\partial t}+\frac{\partial H}{\partial\mathbf{P}}\cdot\frac{\partial\rho}{\partial\mathbf{Q}}-\frac{\partial H}{\partial\mathbf{Q}}\cdot\frac{\partial\rho}{\partial\mathbf{P}} = \frac{\partial\rho}{\partial t}+\{H,\rho\} \tag{10}$$
$$= \frac{\partial\rho}{\partial t}+\{H_1,\rho\}+\frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}\{V_{ij},\rho\} \qquad \text{(Bilinearity of Poisson bracket)}\,. \tag{11}$$

The last term can be re-written by re-indexing and using the symmetry of $V_{ij}=V_{ji}$:

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}\{V_{ij},\rho\} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}\sum_{k=1}^{N}\frac{\partial V_{ij}}{\partial\mathbf{q}_k}\cdot\frac{\partial\rho}{\partial\mathbf{p}_k} = \frac{1}{2}\sum_{i=1}^{N}\sum_{j\neq i}\left[\frac{\partial V_{ij}}{\partial\mathbf{q}_i}\cdot\frac{\partial\rho}{\partial\mathbf{p}_i}+\frac{\partial V_{ij}}{\partial\mathbf{q}_j}\cdot\frac{\partial\rho}{\partial\mathbf{p}_j}\right] = \sum_{i=1}^{N}\frac{\partial\rho}{\partial\mathbf{p}_i}\cdot\sum_{j\neq i}\frac{\partial V_{ij}}{\partial\mathbf{q}_i}\,. \tag{12}$$

Thus, we find the Liouville equation

$$\boxed{\frac{\partial\rho}{\partial t}+\{\rho,H_1\} = \sum_{i=1}^{N}\frac{\partial\rho}{\partial\mathbf{p}_i}\cdot\sum_{j\neq i}\frac{\partial V_{ij}}{\partial\mathbf{q}_i} \qquad \textbf{(Liouville's equation)}} \tag{13}$$

The left-hand side includes the one-body effects, such as advection due to the $\mathbf{P}$ and the flows under U. The right-hand side accounts for transfer of probability due to interactions. The Liouville equation is exact.

2 The BBGKY hierarchy

The Liouville equation (13) for the probability density over the $6N$-dimensional phase space contains way too much information. We are interested in macroscopic quantities, like the average kinetic energy of the gas

$$\text{K.E.} \equiv \left\langle\frac{1}{N}\sum_{i=1}^{N}\frac{p_i^2}{2m}\right\rangle = \frac{1}{m}\langle p_1^2\rangle = \frac{1}{m}\int\prod_{i=1}^{N}d\Gamma_i\,\rho(\mathbf{Q},\mathbf{P};t)p_1^2 \equiv \frac{1}{m}\int d\Gamma_1\,\rho_1(\mathbf{q}_1,\mathbf{p}_1;t)p_1^2\,, \tag{14}$$

where we have used the indistinguishability of the particles, and defined the 1-body probability density as the marginal probability density

$$\rho_1(\mathbf{q}_1,\mathbf{p}_1;t) \equiv \int\prod_{i=2}^{N}d\Gamma_i\,\rho(\mathbf{Q},\mathbf{P};t)\,. \tag{15}$$

Observables like Eq. (14) are one-body properties, which only require ρ_1, which is over a space of much lower dimension. Thus, it is sensible to look for the evolution of ρ_1.

Using the Liouville equation (13), we find

$$\frac{\partial\rho_1}{\partial t} = \int\prod_{i\geq 2}d\Gamma_i\frac{\partial}{\partial t}\rho(\mathbf{Q},\mathbf{P};t) = \int\prod_{i\geq 2}d\Gamma_i\left[\underbrace{\{H_1,\rho\}}_{\equiv 1}+\underbrace{\sum_{j=1}^{N}\frac{\partial\rho}{\partial\mathbf{p}_j}\cdot\sum_{k\neq j}\frac{\partial V_{jk}}{\partial\mathbf{q}_j}}_{\equiv 2}\right]\,. \tag{16}$$

Calculating each term individually, we have

$$\textcircled{1} = \int \prod_{i\geq 2} d\Gamma_i \Bigg[\underbrace{\left(\frac{\partial H_1}{\partial \mathbf{q}_1} \cdot \frac{\partial \rho}{\partial \mathbf{p}_1} - \frac{\partial H_1}{\partial \mathbf{p}_1} \cdot \frac{\partial \rho}{\partial \mathbf{q}_1} \right)}_{\equiv \textcircled{1a}} + \underbrace{\sum_{j\geq 2} \left(\frac{\partial H_1}{\partial \mathbf{q}_j} \cdot \frac{\partial \rho}{\partial \mathbf{p}_j} - \frac{\partial H_1}{\partial \mathbf{p}_j} \cdot \frac{\partial \rho}{\partial \mathbf{q}_j} \right)}_{\equiv \textcircled{1b}} \Bigg] \tag{17}$$

Because $\partial H_1/\partial \mathbf{q}_1$ and $\partial H_1/\partial p_1$ only depend on $\mathbf{q}_1$ and $\mathbf{p}_1$, the integral over $\mathbf{q}_2, \mathbf{q}_3$, etc. and $\mathbf{p}_2, \mathbf{p}_3$, etc. passes through it, and we have

$$\textcircled{1a} = \frac{\partial H_1}{\partial \mathbf{q}_1} \cdot \frac{\partial \rho_1}{\partial \mathbf{p}_1} - \frac{\partial H_1}{\partial \mathbf{p}_1} \cdot \frac{\partial \rho_1}{\partial \mathbf{q}_1} = \{H_1, \rho_1\} \ . \tag{18}$$

For part 1b, we use the fact that $\partial H_1/\partial \mathbf{q}_j$ doesn't depend on $\mathbf{p}_j$, and $\partial H_1/\partial \mathbf{p}_j$ doesn't depend on $\mathbf{q}_j$ to write

$$\textcircled{1b} = \int \prod_{i\geq 2} d\Gamma_i \sum_{j\geq 2} \left[\frac{\partial}{\partial \mathbf{p}_j} \cdot \left(\frac{\partial H_1}{\partial \mathbf{q}_j} \rho \right) - \frac{\partial}{\partial \mathbf{q}_j} \cdot \left(\frac{\partial H_1}{\partial \mathbf{p}_j} \rho \right) \right] = 0 \ , \tag{19}$$

since the integral over a total derivative is zero (assuming there are no boundary terms, which is true for either periodic boundary conditions or a normalizeable ρ in open boundary conditions!).

Term 2, the interaction term, is also simplified by splitting the indices between $j = 1$ and $j > 1$:

$$\textcircled{2} = \int \prod_{i\geq 2} d\Gamma_i \Bigg[\underbrace{\frac{\partial \rho}{\partial \mathbf{p}_1} \cdot \sum_{k\neq 1} \frac{\partial V_{1k}}{\partial \mathbf{q}_1}}_{\equiv \textcircled{2a}} + \underbrace{\sum_{j\geq 2} \frac{\partial \rho}{\partial \mathbf{p}_j} \cdot \sum_{k\neq j} \frac{\partial V_{jk}}{\partial \mathbf{q}_j}}_{\equiv \textcircled{2b}} \Bigg] \tag{20}$$

Term 2a can be simplified using the indistinguishability of particles $k \neq 1$:

$$\textcircled{2a} = \int \prod_{i\geq 2} d\Gamma_i \frac{\partial \rho}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} \equiv (N-1) \int d\Gamma_2 \frac{\partial \rho_2}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} \ , \tag{21}$$

where we have defined the *2-body* probability density

$$\rho_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2; t) \equiv \int \prod_{i\geq 3} d\Gamma_i \rho(\mathbf{Q}, \mathbf{P}; t) \ . \tag{22}$$

Finally, term 2b is zero for the same reason as term 1b (19):

$$\textcircled{2b} = \int \prod_{i\geq 2} d\Gamma_i \sum_{j\geq 2} \frac{\partial}{\partial \mathbf{p}_j} \cdot \left(\rho \sum_{k\neq j} \frac{\partial V_{jk}}{\partial \mathbf{q}_j} \right) = 0 \ . \tag{23}$$

Thus, we find the overall 1-body evolution equation

$$\frac{\partial \rho_1}{\partial t} + \{\rho_1, H_1\} = (N-1) \int d\Gamma_2 \frac{\partial \rho_2}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} \ . \tag{24}$$

This contains much less information than the Liouville equation (13). It is *almost* closed in ρ_1, but has the annoying ρ_2-dependence on the right-hand side. Intuitively, this is because the probability density of a single particle can't be understood without accounting for the joint probability density of it encountering another particle. Unfortunately, $\rho_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2) \neq \rho_1(\mathbf{q}_1, \mathbf{p}_1)\rho_1(\mathbf{q}_2, \mathbf{p}_2)$ since the particles are not independent. For example, for repulsive interactions, $\rho_2(\mathbf{q}, \mathbf{q}, \mathbf{p}, \mathbf{p}') < \rho_1(\mathbf{q}, \mathbf{p})\rho_1(\mathbf{q}, \mathbf{p}')$ since having one particle at location $\mathbf{q}$ makes it less likely to have another particle there.

To find the evolution of ρ_2, we can make a similar calculation to Eqs. (16)-(24). Sparing you the details, the final answer is

$$\frac{\partial \rho_2}{\partial t} + \{\rho_2, H_1 + V_{12}\} = (N-2) \int d\Gamma_3 \left[\frac{\partial \rho_3}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{13}}{\partial \mathbf{q}_1} + \frac{\partial \rho_3}{\partial \mathbf{p}_2} \cdot \frac{\partial V_{23}}{\partial \mathbf{q}_2} \right] \ . \tag{25}$$

The 2-body equation contains dependence on the 3-body density. Likewise, the evolution of the 3-body density will depend on the 4-body density, and so on. This is the BBGKY hierarchy. Because we are only interested in macroscopic, few-body obervables, we must truncate this hierarchy somewhere, using some physically-motivated approximation.

3 The Boltzmann equation

Let's define the number densities

$$f_1(\mathbf{q}_1, \mathbf{p}_1, t) \equiv N\rho_1(\mathbf{q}_1, \mathbf{p}_1; t) \tag{26}$$
$$f_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2, t) \equiv N(N-1)\rho_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2; t) \tag{27}$$
$$\vdots$$
$$f_s(\mathbf{q}_1, \ldots, \mathbf{q}_s, \mathbf{p}_1, \ldots, \mathbf{p}_s, t) \equiv \frac{N!}{(N-s)!}\rho_s(\mathbf{q}_1, \ldots, \mathbf{q}_s, \mathbf{p}_1, \ldots, \mathbf{p}_s; t) \ . \tag{28}$$

These are no longer probability densities. The normalization condition for f_1 is, for example,

$$\int d\Gamma_1 f_1(\mathbf{q}_1, \mathbf{p}_1, t) = N \ . \tag{29}$$

Now let's write out the 2-body equation for $f_2(\mathbf{q}_1, \mathbf{q}_2, \mathbf{p}_1, \mathbf{p}_2)$ explicitly:

$$\dot{f}_2 + \underbrace{\frac{\partial f_2}{\partial \mathbf{q}_1}\cdot\frac{\mathbf{p}_1}{m} + \frac{\partial f_2}{\partial \mathbf{q}_2}\cdot\frac{\mathbf{p}_2}{m}}_{\equiv 1} - \underbrace{\left[\frac{\partial f_2}{\partial \mathbf{p}_1}\cdot\frac{\partial U_1}{\partial \mathbf{q}_1} + \frac{\partial f_2}{\partial \mathbf{p}_2}\cdot\frac{\partial U_2}{\partial \mathbf{q}_2}\right]}_{\equiv 2} - \underbrace{\left[\frac{\partial f_2}{\partial \mathbf{p}_1} - \frac{\partial f_2}{\partial \mathbf{p}_2}\right]\cdot\frac{\partial V_{12}}{\partial \mathbf{q}_1}}_{\equiv 3} = \underbrace{\int d\Gamma_3\left[\frac{\partial f_3}{\partial \mathbf{p}_1}\cdot\frac{\partial V_{13}}{\partial \mathbf{q}_1} + \frac{\partial f_3}{\partial \mathbf{p}_2}\cdot\frac{\partial V_{23}}{\partial \mathbf{q}_2}\right]}_{\equiv 4} \ , \tag{30}$$

where we have used the fact that $\partial V_{12}/\partial \mathbf{q}_2 = -\partial V_{12}/\partial \mathbf{q}_1$.

We will now use dimensional analysis to guess which terms from this equation are the most important. For a gas at room temperature, there are a convenient series of scale separation that make this easy. (This is where the applicability of these calculations to other many situations—e.g. astrophysics—breaks down, since long-range interactions and higher densities mess things up.)

Air molecules at room temperature have typical velocities of $v \approx 10^2 m/s$ and interaction radii of $d \approx 10^{-10}m$. Thus, the time it takes a collision to occur $\tau_c \approx d/v \approx 10^{-12}s$ is very small compared to, say, the time it takes a molecule to cross a box $U(\mathbf{q})$ of length $1m$, $\tau_U \approx L/v \approx 10^{-2}s$. The density of air is also very low: $n \equiv N/V \approx 10^{26}/m^3 \ll 1/d^3$. Thus, the distance a particle typically travels between collisions, ℓ_{MF} or the "mean-free path", is large compared to d. This can be estimated by considering the volume $\ell_{\mathrm{MF}}\pi d^2$ swept out by a particle traveling this distance, and comparing it to the typical volume one must search before encountering a particle, V/N:

$$\ell_{\mathrm{MF}}\pi d^2 \approx \frac{V}{N} \qquad \Longrightarrow \qquad \ell_{\mathrm{MF}} \approx \frac{1}{nd^2} \ . \tag{31}$$

This is given by $\ell_{\mathrm{MF}} \approx 10^{-6}m$. The mean-free time is then given by $\tau_{\mathrm{MF}} = \ell_{\mathrm{MF}}/v \approx 10^{-8}m$.

We have found three processes, each well-separated from the other in terms of length and time-scales:

$$\tau_c \ll \tau_{\mathrm{MF}} \ll \tau_U \ , \qquad d \ll \ell_{\mathrm{MF}} \ll \ell_U \ . \tag{32}$$

These are summarized by the following table:

Process	Length scale	Time scale
Collisions	$d \approx 10^{-10}m$	$\tau_c \approx 10^{-12}s$
Free motion between collisions	$\ell_{\mathrm{MF}} \approx 10^{-6}m$	$\tau_{\mathrm{MF}} \approx 10^{-8}s$
Effects of $U(\mathbf{q})$	$\ell_U \approx 1m$	$\tau_U \approx 10^{-2}s$

The Boltzmann equation, which we will now derive, exploits these two separations of length and time scale.

Now let's return to Eq. (30) and examine it term-by-term. All terms have dimension $T^{-1}N^2L^{-6}$. Let V, U, and KE indicate the energy scales of $V(q)$, $U(\mathbf{q})$, and $p_i^2/2m$ respectively. Also suppose that the system size is comparable to ℓ_U, so that $f_2 \sim (N/\ell_U^3)^2$. Finally, define a new "length scale of interest" $\ell \ll \ell_U$, such that $\partial f_2/\partial \mathbf{q}_i \sim f_2/\ell$. We find the approximate scaling of each term

$$\text{①} \quad \sim \quad \text{KE}\left(\frac{N}{\ell_U^3}\right)^2 \frac{1}{\ell}\frac{1}{mv} \tag{33}$$

$$\text{②} \quad \sim \quad U\frac{1}{\ell_U}\left(\frac{N}{\ell_U^3}\right)^2 \frac{1}{mv} \quad \sim \quad \frac{U}{\text{KE}}\frac{\ell}{\ell_U}\text{①} \quad \ll \quad \text{①} \tag{34}$$

$$\text{③} \quad \sim \quad V\frac{1}{d}\left(\frac{N}{\ell_U^3}\right)^2 \frac{1}{mv} \tag{35}$$

$$\text{④} \quad \sim \quad \int V\frac{1}{d}\left(\frac{N}{\ell_U^3}\right)^3 \sim V d^3 \frac{1}{d}\left(\frac{N}{\ell_U^3}\right)^3 \frac{1}{mv} \quad \sim \quad N\frac{d^3}{\ell_U^3}\text{③} \quad \ll \quad \text{③} \tag{36}$$

We can thus eliminate term ②, since the gradients of the external potential are chosen to be significantly smaller than those of f_2 (and the potential energy U is at most comparable with the kinetic energy). We can also, crucially, eliminate term ④, since it is smaller than term ③ by a factor of $nd^3 \sim 10^{-4} \ll 1$. Since ④ contains all the f_3-dependence, we have thus truncated the BBGKY hierarchy.

We are left with the new equation

$$\dot{f}_2 = \left[\frac{\partial f_2}{\partial \mathbf{p}_1} - \frac{\partial f_2}{\partial \mathbf{p}_2}\right] \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} - \frac{\partial f_2}{\partial \mathbf{q}_1} \cdot \frac{\mathbf{p}_1}{m} - \frac{\partial f_2}{\partial \mathbf{q}_2} \cdot \frac{\mathbf{p}_2}{m} \,, \tag{37}$$

where the $=$ sign should really be an $\approx$ but we will (semi-phenomenologically) pretend the strict equality holds from now on.

Let's simplify $\partial f_2/\partial \mathbf{q}_i$ further. We can change the coordinates $\mathbf{q}_1$, $\mathbf{q}_2$ to $\mathbf{q}_+ \equiv (\mathbf{q}_1 + \mathbf{q}_2)/2$ and $\mathbf{q} \equiv \mathbf{q}_1 - \mathbf{q}_2$, and note that (suppressing the $\mathbf{p}$ dependence)

$$\frac{\partial f_2}{\partial \mathbf{q}_1} = 2\frac{\partial f_2}{\partial \mathbf{q}_+} + \frac{\partial f_2}{\partial \mathbf{q}} \,, \qquad \frac{\partial f_2}{\partial \mathbf{q}_2} = 2\frac{\partial f_2}{\partial \mathbf{q}_+} - \frac{\partial f_2}{\partial \mathbf{q}} \,. \tag{38}$$

Since the gradient f_2 with respect to $\mathbf{q}$ is of the order $1/d$ while variations with respect to $\mathbf{q}_+$ are the inverse of a meso- or macroscopic lengthscale (e.g. $\sim 1/\ell$), we can neglect the $\partial/\partial \mathbf{q}_+$ terms, and approximate

$$\frac{\partial f_2}{\partial \mathbf{q}_1} \approx \frac{\partial f_2}{\partial \mathbf{q}} \,, \qquad \frac{\partial f_2}{\partial \mathbf{q}_2} \approx -\frac{\partial f_2}{\partial \mathbf{q}} \qquad \Longrightarrow \qquad \frac{\partial f_2}{\partial \mathbf{q}_1} \cdot \frac{\mathbf{p}_1}{m} + \frac{\partial f_2}{\partial \mathbf{q}_2} \cdot \frac{\mathbf{p}_2}{m} \approx \frac{\partial f_2}{\partial \mathbf{q}} \cdot \left(\frac{\mathbf{p}_1}{m} - \frac{\mathbf{p}_2}{m}\right) . \tag{39}$$

Return to the 1-body equation (24), which in terms of f_1 and f_2 is given by

$$\frac{\partial f_1}{\partial t} + \{f_1, H_1\} = \int d\Gamma_2 \frac{\partial f_2}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} \equiv \left.\frac{\partial f_1}{\partial t}\right|_{\text{coll.}} . \tag{40}$$

In the steady state, Eqs. (37) and (39) gives us

$$\left[\frac{\partial f_2}{\partial \mathbf{p}_1} - \frac{\partial f_2}{\partial \mathbf{p}_2}\right] \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} = \frac{\partial f_2}{\partial \mathbf{q}} \cdot \left(\frac{\mathbf{p}_1}{m} - \frac{\mathbf{p}_2}{m}\right) \tag{41}$$

$$\Longrightarrow \int d\Gamma_2 \left[\frac{\partial f_2}{\partial \mathbf{p}_1} - \frac{\partial f_2}{\partial \mathbf{p}_2}\right] \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} = \int d\Gamma_2 \frac{\partial f_2}{\partial \mathbf{p}_1} \cdot \frac{\partial V_{12}}{\partial \mathbf{q}_1} = \int d\Gamma_2 \frac{\partial f_2}{\partial \mathbf{q}} \cdot \left(\frac{\mathbf{p}_1}{m} - \frac{\mathbf{p}_2}{m}\right) . \tag{42}$$

The first equality in Eq. (42) is obtained by noting that the second term is a total derivative in $\mathbf{p}_2$, which is integrated over. Thus, the second equality of Eq. (42) allows us to replace the right-hand side of Eq. (40). Also defining $\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1$ so that the integral is over the relative momentum of coordinate 2 in coordinate 1's frame, we find

$$\left.\frac{\partial f_1}{\partial t}\right|_{\text{coll.}} = -\frac{1}{m}\int d^3\mathbf{q} d^3\mathbf{p} \mathbf{p} \cdot \frac{\partial f_2}{\partial \mathbf{q}}(\mathbf{q}_1, \mathbf{q}_1 + \mathbf{q}, \mathbf{p}_1, \mathbf{p}_1 + \mathbf{p}) \,. \tag{43}$$

Keep in mind that we have made the replacement $\mathbf{q} = \mathbf{q}_1 - \mathbf{q}_2$ and $\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1$.

[End of recitation 4]

Can we simplify the collision term (43) even more? Note that for any function $A(\mathbf{q}, \mathbf{p})$, we can write

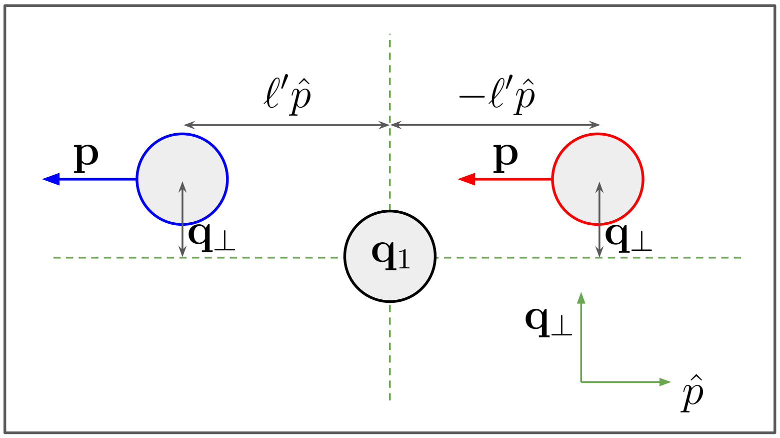
$$\int d^3\mathbf{p} d^3\mathbf{q} \, \mathbf{p} \cdot \frac{\partial A(\mathbf{q}, \mathbf{p})}{\partial \mathbf{q}} = \int d^3\mathbf{p} d^2\mathbf{q}_\perp \, |\mathbf{p}| \Big[A(\mathbf{q}_\perp + \hat{p}\ell', \mathbf{p}) - A(\mathbf{q}_\perp - \hat{p}\ell', \mathbf{p})\Big] \,, \tag{44}$$

where $\ell' \to \infty$, $\hat{\mathbf{p}} = \mathbf{p}/|\mathbf{p}|$, and $\mathbf{q}_\perp \cdot \hat{p} = 0$. That is, we have decomposed the $\mathbf{q}$ space into a plane $\mathbf{q}_\perp$ perpendicular to $\mathbf{p}$, and an axis parallel to $\mathbf{p}$, which we have integrated over. We can do the same with Eq. (43) to find

$$\left.\frac{\partial f_1}{\partial t}\right|_{\text{coll.}} = -\int d^3\mathbf{p} d^2\mathbf{q}_\perp \frac{|\mathbf{p}|}{m}\Big[f_2\big(\mathbf{q}_1, \mathbf{q}_1 - (\mathbf{q}_\perp + \hat{p}\ell'), \mathbf{p}_1, \mathbf{p}_1 + \mathbf{p}, t\big) - f_2\big(\mathbf{q}_1, \mathbf{q}_1 - (\mathbf{q}_\perp - \hat{p}\ell'), \mathbf{p}_1, \mathbf{p}_1 + \mathbf{p}, t\big)\Big]\,. \tag{45}$$

In principle, we must take $\ell' \to \infty$ since the limits of the $\mathbf{q}$ integral is infinity, but we don't actually need to because f_2 only varies with respect to $\mathbf{q}_1 - \mathbf{q}_2$ on a scale of d. Thus, as long as $\ell' \gg d$, we are fine.

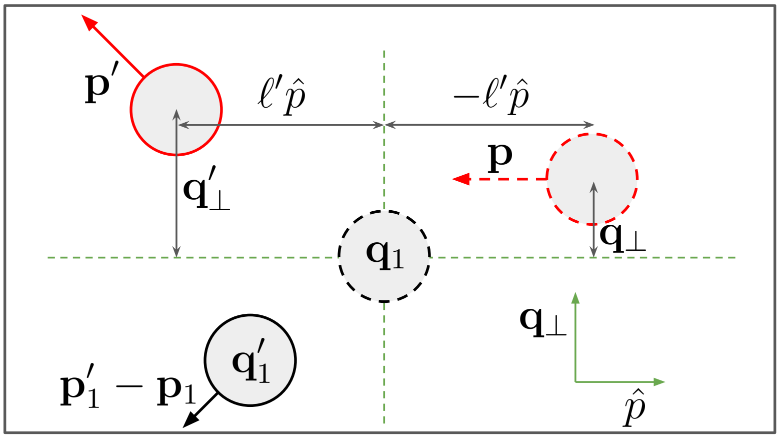
This result, and its relation to collision kinematics, is illustrated below. Working in a frame where $\mathbf{p}_1 = 0$, our integral over $\hat{p}$ results in the difference of two terms: the probability of the blue and black particles' configuration, minus the probability of the red and black particles' configuration.



We can now also note that we can make the spatial coordinates in each term equal using "streaming". (This is where the talk of collisions and scattering really starts to make sense.) Because probability is constant along a trajectory, we can write

$$f_2\big(\mathbf{q}_1, \mathbf{q}_1 - (\mathbf{q}_\perp - \hat{p}\ell'), \mathbf{p}_1, \mathbf{p}_1 + \mathbf{p}, t\big) = f_2\big(\mathbf{q}_1', \mathbf{q}_1' - (\mathbf{q}_\perp' + \hat{p}\ell'), \mathbf{p}_1', \mathbf{p}_1' + \mathbf{p}', t\big) \tag{46}$$

where two particles starting at $\mathbf{q}_1$, $\mathbf{q}_1 - (\mathbf{q}_\perp - \hat{p}\ell')$ with momenta $\mathbf{p}_1$, $\mathbf{p}_1 + \mathbf{p}$ end up at $\mathbf{q}_1'$, $\mathbf{q}_1 - (\mathbf{q}_\perp' + \hat{p}\ell')$ with momenta $\mathbf{p}_1'$, $\mathbf{p}_1' + \mathbf{p}'$. (You can check that we have enough degrees of freedom for some such $\mathbf{q}_1'$, $\mathbf{q}_\perp'$, $\mathbf{p}_1'$, and $\mathbf{p}'$ to exist.) This is illustrated below: Again, working in the frame where $\mathbf{p}_1 = 0$, we have replace the probability of the red/black particle configuration (now drawn with dashed borders) with the probability of the evolved system, at $\mathbf{q}_1'$, $\mathbf{q}_1 - (\mathbf{q}_\perp + \hat{p}\ell')$ with new momenta $\mathbf{p}_1'$, $\mathbf{p}_1 + \mathbf{p}'$:



However, because f_2 doesn't depend strongly on the center of mass $\mathbf{q}_1 + \mathbf{q}_2$, we can use $\mathbf{q}_1' \approx \mathbf{q}_1$. In fact, we can also use

$\mathbf{q}_1 - (\mathbf{q}_\perp - \hat{p}\ell') \approx \mathbf{q}_1$, as long as $\ell' \ll \ell$ (which is possible since $d \ll \ell$).[1] All in all, we may write

$$\left.\frac{\partial f_1}{\partial t}\right|_{\text{coll.}} = -\int d^3\mathbf{p}d^2\mathbf{q}_\perp \frac{|\mathbf{p}|}{m}\Big[f_2(\mathbf{q}_1, \mathbf{q}_1, \mathbf{p}_1, \mathbf{p}_1 + \mathbf{p}, t) - f_2(\mathbf{q}_1, \mathbf{q}_1, \mathbf{p}_1', \mathbf{p}_1' + \mathbf{p}', t)\Big] \tag{47}$$

$$= -\int d^3\mathbf{p}_2 d^2\mathbf{q}_\perp \frac{|\mathbf{p}|}{m}\Big[f_2(\mathbf{q}_1, \mathbf{q}_1, \mathbf{p}_1, \mathbf{p}_2, t) - f_2(\mathbf{q}_1, \mathbf{q}_1, \mathbf{p}_1', \mathbf{p}_2', t)\Big] , \tag{48}$$

for some $\mathbf{p}_1'$, $\mathbf{p}' = \mathbf{p}_2' - \mathbf{p}_1'$ which are related to $\mathbf{p}_1$ and $\mathbf{p} = \mathbf{p}_2 - \mathbf{p}_1$ through the equations of motion, in a way that depends on $\mathbf{q}_\perp$ and ℓ'. Note that we have changed coordinates back to $\mathbf{p}_2 = \mathbf{p} + \mathbf{p}_1$.

Finally, to turn this into a closed equation of f_1, we make the *molecular chaos* assumption, that the particles are uncorrelated. That is, we assume

$$f_2(\mathbf{q}_1, \mathbf{q}_1, \mathbf{p}_1, \mathbf{p}_2, t) \approx f_1(\mathbf{q}_1, \mathbf{p}_1, t) f_1(\mathbf{q}_1, \mathbf{p}_2, t) \tag{49}$$

$$f_2(\mathbf{q}_1, \mathbf{q}_1, \mathbf{p}_1', \mathbf{p}_2', t) \approx f_1(\mathbf{q}_1, \mathbf{p}_1', t) f_1(\mathbf{q}_1, \mathbf{p}_2', t) . \tag{50}$$

Crucially, we assume that particles are uncorrelated both pre-collision *and* post-collision, something that isn't necessarily true.

Thus, we are left with the Boltzmann equation

$$\boxed{\frac{\partial f_1}{\partial t} + \{f_1, H_1\} = \frac{1}{m}\int d^2\mathbf{q}_\perp d^3\mathbf{p}_2 |\mathbf{p}_1 - \mathbf{p}_2| \big[f_1(\mathbf{q}_1, \mathbf{p}_1', t) f_1(\mathbf{q}_1, \mathbf{p}_2', t) - f_1(\mathbf{q}_1, \mathbf{p}_1, t) f_1(\mathbf{q}_1, \mathbf{p}_2, t)\big] .} \tag{51}$$

This is finally a closed equation for f_1. The right-hand side describes how collisions between particles co-located near $\mathbf{q}_1$ with "incidence vector" $\mathbf{q}_\perp$ transfer momentum from $(\mathbf{p}_1, \mathbf{p}_2)$ to $(\mathbf{p}_1', \mathbf{p}_2')$, a process that occurs with a weight $d^2\mathbf{q}_\perp |\mathbf{p}_1 - \mathbf{p}_2|/m$. The final momenta $(\mathbf{p}_1', \mathbf{p}_2')$ are deterministically related to $\mathbf{q}_\perp$ and the initial momenta, and the dependence requires knowledge of V. (Note that we could also make a variable change to the $d^2\mathbf{q}_\perp$ integral to re-write Eq. (51) in terms of scattering cross sections. In this case, there is still an implicit relationship between $(\mathbf{p}_1', \mathbf{p}_2')$ and $(\mathbf{p}_1, \mathbf{p}_2)$ which is determined by the form of the potential V.)

After all this hand-waving, you may not believe Eq. (51). If you like, you can also think of it as a phenomenological equation: the simplest equation, to this order in f_1, that describes the effect of collisions while also respecting the symmetries of the system. Then, the scattering cross section and the relationship between the $\mathbf{p}_i'$ and the $\mathbf{p}_i$ can be thought of as "free parameters" constrained only by the symmetries of the dynamics. However, it is also true that the steps made in this derivation can be justified more rigorously. In fact, mathematicians seem to have recently proven that Eq. (51) is a valid description of the dynamics of dilute hard spheres.[2] With a physicist's appeal to universality, we can comfortably extend this to any dilute gas with short-ranged interactions.

In celebration of our newly closed equation (51), we will henceforth replace f_1 with f in our notation, understanding that f always corresponds to the 1-body distribution.

4 The H-theorem

We started with time-reversible dynamics (Eq. (13)). Is the Boltzmann equation (51) time-reversible as well? The H-theorem tells us that it is not.

Theorem 4.1: The H-theorem

If $f(\mathbf{q}, \mathbf{p}, t)$ satisfies the Boltzmann equation (51), then $dH/dt \leq 0$, for

$$H(t) = \int d^3\mathbf{q} d^3\mathbf{p} f(\mathbf{q}, \mathbf{p}, t) \ln f(\mathbf{q}, \mathbf{p}, t) . \tag{52}$$

You will recognize the right-hand side to be proportional to the negative entropy of the system. Thus, the H-theorem tells us that entropy always increases. Now, let's go through the proof.

[1] Note that, because the momenta change significantly during collisions, we couldn't have just made the positions equal in Eq. (45). (This would clearly result in the collision term being zero!) These approximations and substitutions basically amount to the fact that the momenta "teleport" a significant distance during collision, which are themselves not spatially-resolved.

[2] Y. Deng et al., *Long time derivation of the Boltzmann equation from hard sphere dynamics*, arXiv:2408.07818 [math], July 2025.

Proof 4.1: Proof of H-theorem.

Use the shorthand notation $\int d^3\mathbf{q}d^3\mathbf{p} \equiv \int_{\mathbf{q},\mathbf{p}}$. We can explicitly write the time evolution of $H(t)$ using the Boltzmann equation as

$$\frac{dH}{dt} = \int_{\mathbf{q},\mathbf{p}} \frac{\partial f}{\partial t}\big(\ln f + 1\big) = \int_{\mathbf{q},\mathbf{p}} \frac{\partial f}{\partial t} \ln f = \int_{\mathbf{q},\mathbf{p}} \Bigg(\underbrace{\{H_1, f\}}_{\equiv ①} + \underbrace{\frac{\partial f}{\partial t}\bigg|_{\text{coll.}}}_{\equiv ②}\Bigg) \ln f \tag{53}$$

where

$$① = \int_{\mathbf{q},\mathbf{p}} \{H_1, f\} \ln f = \int_{\mathbf{q},\mathbf{p}} \left(\frac{\partial U}{\partial \mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{p}} - \frac{\mathbf{p}}{m} \cdot \frac{\partial f}{\partial \mathbf{q}}\right) \ln f = 0 \qquad \text{(IBP)} \tag{54}$$

and

$$② = \int_{\mathbf{q},\mathbf{p}} \frac{\partial f}{\partial t}\bigg|_{\text{coll.}} \ln f \tag{55}$$

$$= \int_{\mathbf{q},\mathbf{p}_1} \int d^3\mathbf{p}_2 d^2\vec{b} \frac{|\mathbf{p}_1 - \mathbf{p}_2|}{m} \Big[f(\mathbf{q},\mathbf{p}_1')f(\mathbf{q},\mathbf{p}_2') - f(\mathbf{q},\mathbf{p}_1)f(\mathbf{q},\mathbf{p}_2)\Big] \ln f(\mathbf{q}_1,\mathbf{p}_1)\ . \tag{56}$$

Note that for an arbitrary function $A(\mathbf{p}_1,\mathbf{p}_2)$,

$$\int d^3\mathbf{p}_1 d^3\mathbf{p}_2 A(\mathbf{p}_1,\mathbf{p}_2) = \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 A(\mathbf{p}_2,\mathbf{p}_1) = \int d^3\mathbf{p}_1 d^3\mathbf{p}_2 \frac{1}{2}\big[A(\mathbf{p}_1,\mathbf{p}_2) + A(\mathbf{p}_2,\mathbf{p}_1)\big]\ . \tag{57}$$

Thus, we can apply this symmetrization to ② to find, suppressing all $\mathbf{q}$-dependence since everything occurs at the same spatial coordinate,

$$② = \int_{\mathbf{q},\mathbf{p}_1,\mathbf{p}_2,\vec{b}} \frac{|\mathbf{p}_1 - \mathbf{p}_2|}{m}\frac{1}{2}\bigg\{\Big[f(\mathbf{p}_1')f(\mathbf{p}_2') - f(\mathbf{p}_1)f(\mathbf{p}_2)\Big]\ln f(\mathbf{p}_1) + \Big[f(\mathbf{p}_2')f(\mathbf{p}_1') - f(\mathbf{p}_2)f(\mathbf{p}_1)\Big]\ln f(\mathbf{p}_2)\bigg\} \tag{58}$$

$$= \int_{\mathbf{q},\mathbf{p}_1,\mathbf{p}_2,\vec{b}} \frac{|\mathbf{p}_1 - \mathbf{p}_2|}{2m}\Big[f(\mathbf{p}_2')f(\mathbf{p}_1') - f(\mathbf{p}_2)f(\mathbf{p}_1)\Big]\ln\big[f(\mathbf{p}_1)f(\mathbf{p}_2)\big]\ . \tag{59}$$

Now, it is useful to keep in mind that $\mathbf{p}_1'$ and $\mathbf{p}_2'$ are really functions of $\vec{b}$, $\mathbf{p}_1$, and $\mathbf{p}_2$. Moreover, this functional relationship is invertible and has unit Jacobian. Recall also that $|\mathbf{p}_1' - \mathbf{p}_2'| = |\mathbf{p}_1 - \mathbf{p}_2|$. Then, we can use the identity (for an invertible $\mathbf{u}(\mathbf{x})$)

$$\int d^d\mathbf{x} F\big(\mathbf{u}(\mathbf{x})\big) = \int \frac{d^d\mathbf{u}}{|\det \mathcal{J}_{\mathbf{u}}|} F(\mathbf{u}) \tag{60}$$

to make an additional symmetrization

$$② = \int_{\mathbf{q},\mathbf{p}_1,\mathbf{p}_2,\vec{b}} \frac{|\mathbf{p}_1 - \mathbf{p}_2|}{2m}\frac{1}{2}\bigg\{\Big[f(\mathbf{p}_2')f(\mathbf{p}_1') - f(\mathbf{p}_2)f(\mathbf{p}_1)\Big]\ln\big[f(\mathbf{p}_1)f(\mathbf{p}_2)\big] \tag{61}$$

$$+ \Big[f(\mathbf{p}_2)f(\mathbf{p}_1) - f(\mathbf{p}_2')f(\mathbf{p}_1')\Big]\ln\big[f(\mathbf{p}_1')f(\mathbf{p}_2')\big]\bigg\}\ , \tag{62}$$

where in the second term $\mathbf{p}_1$ and $\mathbf{p}_2$ are understood as functions of $\mathbf{p}_1'$, $\mathbf{p}_2'$, and $\vec{b}$. Thus, we find

$$\frac{dH}{dt} = ② = -\int_{\mathbf{q},\mathbf{p}_1,\mathbf{p}_2,\vec{b}} \frac{|\mathbf{p}_1 - \mathbf{p}_2|}{4m} \underbrace{\Big[f(\mathbf{p}_2')f(\mathbf{p}_1') - f(\mathbf{p}_2)f(\mathbf{p}_1)\Big]}_{\equiv ⓐ} \underbrace{\Big[\ln\big[f(\mathbf{p}_1')f(\mathbf{p}_2')\big] - \ln\big[f(\mathbf{p}_1)f(\mathbf{p}_2)\big]\Big]}_{\equiv ⓑ}\ . \tag{63}$$

When $f(\mathbf{p}_2')f(\mathbf{p}_1') > f(\mathbf{p}_2)f(\mathbf{p}_1)$, term ⓐ is positive, but so is term ⓑ. When $f(\mathbf{p}_2')f(\mathbf{p}_1') < f(\mathbf{p}_2)f(\mathbf{p}_1)$, term ⓐ is negative, but so is term ⓑ. All in all, we see that $dH/dt \leq 0$.

Thus the reversible microscopic dynamics exhibit an apparent macroscopic irreversibility. Two questions: (1) why is this the case, and (2) how can the Boltzmann equation (51) still be an accurate description of the dynamics?

1. **How is the Boltzmann equation irreversible?** We have made many approximations to get to Eq. (51), but not all of them manifestly violate time-reversibility. However, we can identify one culprit: in the molecular chaos approximation (49)-(50), we have assumed that particles are just as uncorrelated before a collision as they are after. In reversible dynamics, this isn't possible.

2. **How can an irreversible equation accurately describe reversible dynamics?** The answer to this question involves coarse-graining. While it is, in principle, possible for all the gas in a room to move to one side of the room, understanding this process would require detailed knowledge of all the microscopic degrees of freedom and all of their correlations. We are not interested in such a description, and it isn't needed to describe the most probable macroscopic behavior of the system.

   Another more mathematical way to understand this is as follows: consider a system with many states, some more likely than others. (E.g. some have multiple copies.) Imagine that it is possible to transition between any two states, and that the dynamics of these transitions satisfy time-reversal symmetry.[3] In search of a lower-dimensional description of our system, we may disregard some of the transitions. This in general ruins the time-reversal symmetry of the system. However, if we eliminate the more unlikely transitions, the description will be accurate enough to describe the overall behavior of the system. One way to "keep the most likely transitions" is to favor those which increase the degeneracy of states. The result is that the entropy of the system increases, i.e. $\dot{H} \leq 0$. This is, in principle, what we have done in our derivation of the Boltzmann equation, although it isn't so transparent.

4.1 Consequences: the local equilibrium distribution

Because H is monotonically decreasing, its steady state is constant. We can thus deduce the steady state of the system by requiring $\dot{H} = 0$. In fact, we can make the following equivalence:

$$\dot{H} = 0 \quad \Leftrightarrow \quad f(\mathbf{q}_1, \mathbf{p}_1) f(\mathbf{q}_1, \mathbf{p}_2) = f(\mathbf{q}_1, \mathbf{p}_1') f(\mathbf{q}_1, \mathbf{p}_2') \quad \Leftrightarrow \quad \ln f(\mathbf{q}_1, \mathbf{p}_1) + \ln f(\mathbf{q}_1, \mathbf{p}_2) = \ln f(\mathbf{q}_1, \mathbf{p}_1') + \ln f(\mathbf{q}_1, \mathbf{p}_2') \; . \tag{64}$$

Because $(\mathbf{p}_1, \mathbf{p}_2)$ and $(\mathbf{p}_1', \mathbf{p}_2')$ are the momenta before and after a collision, this implies that $\ln f(\mathbf{q}_1, \mathbf{p}_1) + \ln f(\mathbf{q}_1, \mathbf{p}_2)$ is the same value before and after collisions. Thus, $\ln f(\mathbf{q}, \mathbf{p})$ must be a linear combination of quantities that are conserved by the collisions. We can identify 3 such quantities:

1. Particle number: $1 + 1 = 1 + 1$
2. Momentum: $\mathbf{p}_1 + \mathbf{p}_2 = \mathbf{p}_1' + \mathbf{p}_2'$
3. Kinetic energy: $|\mathbf{p}_1|^2 + |\mathbf{p}_2|^2 = |\mathbf{p}_1'|^2 + |\mathbf{p}_2'|^2$.

We can thus write the "local equilibrium" form of f:

$$f_{\text{LEQ}}(\mathbf{q}, \mathbf{p}) = \gamma(\mathbf{q}) \exp\left[-\boldsymbol{\alpha}(\mathbf{q}) \cdot \mathbf{p} - \beta(\mathbf{q}) \left(\frac{|\mathbf{p}|^2}{2m} + U(\mathbf{q}) \right) \right] . \tag{65}$$

We have exercised our freedom of choice of the $\mathbf{q}$-dependent prefactors to write it this way. You can check that

$$\left. \frac{\partial f_{\text{LEQ}}}{\partial t} \right|_{\text{coll.}} = 0 \; . \tag{66}$$

However, $\dot{f}_{\text{LEQ}} \neq 0$ in general: it must also have zero $\{H_1, f\}$, which requires particular choices of $\gamma, \boldsymbol{\alpha}, \beta$.

Eq. (65) represents the "local equilibrium" reached on timescales $\sim \tau_{\text{MF}}$. Because momenta and kinetic energy are exchanged in collisions, after a few collisions have occurred, their probability distribution is roughly the same across all particles in a local region. Once this has occurred, the system undergoes a slower relaxation to global equilibrium. This time period is governed by what we call the system's "hydrodynamic" description.

Another way to justify this calculation is through a perturbation in the small parameter τ_{MF}/τ_U. Recall that τ_U is an extrinsic timescale, describing the evolution of the system due to the external potential U. The one-body term $\{H_1, f\}$ in $\partial_t f$ scales like $1/\tau_U$. Moreover, the collision term scales like $1/\tau_{\text{MF}}$. Thus we can define the non-dimensionalized operators

$$\mathcal{L}[f] \equiv \tau_U \{H_1, f\} \tag{67}$$

$$\mathcal{C}[f, g] \equiv \tau_{\text{MF}} \int d^3\tilde{\mathbf{p}} d^2\mathbf{q}_\perp \left[f(\mathbf{q}, \mathbf{p}) g(\mathbf{q}, \tilde{\mathbf{p}}) - f(\mathbf{q}, \mathbf{p}') g(\mathbf{q}, \tilde{\mathbf{p}}') \right] \tag{68}$$

[3] We would say, for states A and B with probabilities $P(A)$ and $P(B)$ and transition rates $P(A \to B)$ and $P(B \to A)$, that $P(A)P(A \to B) = P(B)P(B \to A)$.

so that

$$\frac{\partial f}{\partial t} = \frac{1}{\tau_U}\mathcal{L}[f] + \frac{1}{\tau_{\text{MF}}}\mathcal{C}[f,f] \;, \quad \text{so in steady state} \tag{69}$$

$$\frac{\partial f}{\partial t} = 0 \quad \Longrightarrow \quad \varepsilon\mathcal{L}[f] + \mathcal{C}[f,f] = 0 \;, \quad \text{where} \quad \varepsilon \equiv \frac{\tau_{\text{MF}}}{\tau_U} \;. \tag{70}$$

We can then write the solution of the Boltzmann equation as

$$f(\mathbf{q},\mathbf{p},t) = f^{(0)}(\mathbf{q},\mathbf{p},t) + \varepsilon f^{(1)}(\mathbf{q},\mathbf{p},t) + \varepsilon^2 f^{(2)}(\mathbf{q},\mathbf{p},t) + \dots \tag{71}$$

where the terms are determined by equations like

$$\mathcal{C}[f^{(0)}, f^{(0)}] = 0 \tag{72}$$

$$\mathcal{L}[f^{(0)}] + \mathcal{C}[f^{(0)}, f^{(1)}] + \mathcal{C}[f^{(1)}, f^{(0)}] = 0 \tag{73}$$

$$\dots . \tag{74}$$

The order-ε^0 solution is $f^{(0)} = f_{\text{LEQ}}$, given in Eq. (65). We will examine the consequences of this perturbation scheme in the next section. After making some definitions, we will look at how this 0th-order solution $f^{(0)}$ behaves in Sec. 5.2. Then in Sec. 5.3, we will find an approximate solution for $f^{(1)}$ and investigate its behavior.

5 Hydrodynamics

5.1 Conserved quantities

The long-timescale evolution of the system is governed by what we call "slow fields", fields which take a long time to relax ($\tau \gg \tau_{\text{MF}}$). In this system, it can be argued that the only such fields are the conserved quantities. These are given by the particle density n, local velocity $\mathbf{u}$, and local "comoving" kinetic energy ϵ, defined as follows:

$$n(\mathbf{q},t) \equiv \int d^3\mathbf{p} f(\mathbf{q},\mathbf{p},t) \tag{75}$$

$$\mathbf{u}(\mathbf{q},t) \equiv \frac{1}{n(\mathbf{q},t)} \int d^3\mathbf{p}\, \frac{\mathbf{p}}{m} f(\mathbf{q},\mathbf{p},t) \tag{76}$$

$$\epsilon(\mathbf{q},t) \equiv \frac{1}{n(\mathbf{q},t)} \int d^3\mathbf{p}\, \frac{m}{2}\left|\frac{\mathbf{p}}{m} - \mathbf{u}(\mathbf{q},t)\right|^2 f(\mathbf{q},\mathbf{p},t) \;. \tag{77}$$

We also define the local averaging operation $\langle \cdot \rangle$: for an observable $\mathcal{O}(\mathbf{q},\mathbf{p},t)$, we write

$$\langle \mathcal{O}(\mathbf{q},\mathbf{p},t)\rangle = \frac{1}{n(\mathbf{q},t)} \int d^3\mathbf{p}\mathcal{O}(\mathbf{q},\mathbf{p},t) f(\mathbf{q},\mathbf{p},t) \;. \tag{78}$$

Thus, $n(\mathbf{q},t) = n(\mathbf{q},t)\langle 1\rangle$, $\mathbf{u}(\mathbf{q},t) = n(\mathbf{q},t)\langle \mathbf{p}/m\rangle$, and $\epsilon(\mathbf{q},t) = n(\mathbf{q},t)\langle \frac{m}{2}|\mathbf{u} - \mathbf{p}/2m|^2\rangle$.

The dynamics of these quantities are relatively simple because they are conserved by collisions. As a consequence, we have the following lemma:

Lemma 5.1: Collision integral of conserved quantities is zero

For a quantity $\chi(\mathbf{q},\mathbf{p})$ which is conserved during a collision, i.e.

$$\chi(\mathbf{q},\mathbf{p}_1,t) + \chi(\mathbf{q},\mathbf{p}_2,t) = \chi(\mathbf{q},\mathbf{p}_1',t) + \chi(\mathbf{q},\mathbf{p}_2',t) \;, \tag{79}$$

its flux due to collisions is zero:

$$J_\chi(\mathbf{q},t) \equiv \int d^3\mathbf{p}\chi(\mathbf{q},\mathbf{p},t) \left.\frac{\partial f}{\partial t}\right|_{\text{coll.}} = 0 \;. \tag{80}$$

We will skip the proof, which is straightforward algebra. As a consequence, we can compute the evolution of χ-density:

$$\partial_t(n\langle\chi\rangle) = \frac{\partial}{\partial t}\left[\int d^3\mathbf{p}\chi(\mathbf{q},\mathbf{p},t)f(\mathbf{q},\mathbf{p},t)\right] = \int d^3\mathbf{p}\big[\chi\partial_t f + f + \partial_t\chi \tag{81}$$

$$= \int d^3\mathbf{p}\left[\chi\{H_1,f\} + \chi\frac{\partial f}{\partial t}\Big|_{\text{coll.}} + f\partial_t\chi\right] = \int d^3\mathbf{p}\big[\chi\{H_1,f\} + f\partial_t\chi\big] \tag{82}$$

$$= \int d^3\mathbf{p}\left[\chi\frac{\partial U}{\partial\mathbf{q}}\cdot\frac{\partial f}{\partial\mathbf{p}} - \chi\frac{\mathbf{p}}{m}\cdot\frac{\partial f}{\partial\mathbf{q}} + f\partial_t\chi\right] \tag{83}$$

$$= \int d^3\mathbf{p}\left[-f\frac{\partial U}{\partial\mathbf{q}}\cdot\frac{\partial\chi}{\partial\mathbf{p}} - \frac{\mathbf{p}}{m}\cdot\frac{\partial}{\partial\mathbf{q}}(\chi f) + f\frac{\mathbf{p}}{m}\cdot\frac{\partial\chi}{\partial\mathbf{q}} + f\partial_t\chi\right] \qquad \text{(IBP)}\ . \tag{84}$$

As a result, we find

$$\partial_t(n\langle\chi\rangle) = -\frac{\partial}{\partial\mathbf{q}}\cdot\left(n\left\langle\frac{\mathbf{p}}{m}\chi\right\rangle\right) - n\frac{\partial U}{\partial\mathbf{q}}\cdot\left\langle\frac{\partial\chi}{\partial\mathbf{p}}\right\rangle + n\left\langle\frac{\mathbf{p}}{m}\cdot\frac{\partial}{\partial\mathbf{q}}\chi\right\rangle + n\langle\partial_t\chi\rangle\ . \tag{85}$$

Now we will use Eq. (85) to find the evolution of n, $\mathbf{u}$, and ϵ.

5.1.1 Evolution of $n(\mathbf{q},t)$

Letting $\chi = 1$, Eq. (85) becomes

$$\partial_t n = -\partial_\alpha(nu_\alpha)\ . \tag{86}$$

From now on, repeated greek indices are summed over.

5.1.2 Evolution of $\mathbf{u}(\mathbf{q},t)$

Letting $\chi = \mathbf{p}/m$ so that $\langle\chi\rangle = \mathbf{u}$, Eq. (85) becomes

$$\partial_t(nu_\alpha) = -\frac{1}{m^2}\partial_\beta(n\langle p_\alpha p_\beta\rangle) - \frac{n}{m}\partial_\alpha U \tag{87}$$

$$\equiv -\partial_\beta\left(\frac{P_{\alpha\beta}}{m} + nu_\alpha u_\beta\right) - \frac{n}{m}\partial_\alpha U \tag{88}$$

$$\implies n\partial_t u_\alpha + u_\alpha\underbrace{\partial_t n}_{=-\partial_\beta(nu_\beta)} = -\frac{1}{m}\partial_\beta P_{\alpha\beta} - u_\alpha\partial_\beta(nu_\beta) - nu_\beta\partial_\beta u_\alpha - \frac{n}{m}\partial_\alpha U \tag{89}$$

$$\implies \partial_t u_\alpha = -\frac{1}{nm}\partial_\beta P_{\alpha\beta} - u_\beta\partial_\beta u_\alpha - \frac{1}{m}\partial_\alpha U\ . \tag{90}$$

We have defined

$$P_{\alpha\beta} = mn\left\langle\left(\frac{p_\alpha}{m} - u_\alpha\right)\left(\frac{p_\beta}{m} - u_\beta\right)\right\rangle \tag{91}$$

Defining the material derivative

$$D_t A(\mathbf{q},t) \equiv \partial_t A + u_\beta\partial_\beta A\ , \tag{92}$$

we find

$$D_t u_\alpha = -\frac{1}{nm}\partial_\beta P_{\alpha\beta} - \frac{1}{m}\partial_\alpha U\ . \tag{93}$$

This is just Newton's $ma = F$ law written in fluid form. The left is the acceleration, and the right is the force due to internal and external energy, respectively.

5.1.3 Evolution of $\epsilon(\mathbf{q}, t)$

Finally, use Eq. (85) with $\chi = \frac{m}{2}\left|\mathbf{u} - \frac{\mathbf{p}}{m}\right|^2$ so that $\langle\chi\rangle = \epsilon$, to write

$$\partial_t(n\epsilon) = n\partial_t\epsilon + \epsilon \underbrace{\partial_t n}_{=-\partial_\alpha(nu_\alpha)} \tag{94}$$

$$= -\partial_\alpha\left[n\left\langle\frac{p_\alpha}{m}\frac{m}{2}\left(\frac{p_\beta}{m} - u_\beta\right)\left(\frac{p_\beta}{m} - u_\beta\right)\right\rangle\right] - n\partial_\alpha U \underbrace{\langle p_\alpha/m - u_\alpha\rangle}_{=0} + n\left\langle\frac{p_\alpha}{m} m\left(u_\beta - \frac{p_\beta}{m}\right)\partial_\alpha u_\beta\right\rangle - nm\underbrace{\left\langle\frac{p_\alpha}{m} - u_\alpha\right\rangle}_{=0}\partial_t u_\alpha \tag{95}$$

$$\equiv -\partial_\alpha h_\alpha - \partial_\alpha(nu_\alpha\epsilon) + nm\left\langle\left(\frac{p_\alpha}{m} - u_\alpha\right)\left(u_\beta - \frac{p_\beta}{m}\right)\partial_\alpha u_\beta\right\rangle + nm\left\langle u_\alpha\underbrace{\left(u_\beta - \frac{p_\beta}{m}\right)}_{=0}\partial_\alpha u_\beta\right\rangle \tag{96}$$

$$= -\partial_\alpha h_\alpha - \partial_\alpha(nu_\alpha\epsilon) - P_{\alpha\beta}\partial_\alpha u_\beta \tag{97}$$

where we have defined

$$h_\alpha \equiv \frac{nm}{2}\left\langle\left(\frac{p_\alpha}{m} - u_\alpha\right)\left(\frac{p_\beta}{m} - u_\beta\right)\left(\frac{p_\beta}{m} - u_\beta\right)\right\rangle . \tag{98}$$

This implies that

$$n\partial_t\epsilon = -\partial_\alpha h_\alpha - nu_\alpha\partial_\alpha\epsilon - P_{\alpha\beta}u_{\alpha\beta} \tag{99}$$

$$\implies D_t\epsilon = -\frac{1}{n}\left[\partial_\alpha h_\alpha + P_{\alpha\beta}u_{\alpha\beta}\right] . \tag{100}$$

We have also defined the rate of strain tensor (using the symmetry of $P_{\alpha\beta}$ and the fact that α and β are summed-over dummy indices)

$$u_{\alpha\beta} \equiv \frac{1}{2}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha\right) . \tag{101}$$

Eqs. (86), (93), and (100) thus constitute the hydrodynamic description of our system. Now, we will study their steady-state solutions.

[END OF RECITATION 5]

Let's repeat the hydrodynamic equations (86), (93), and (100) for clarity:

$$D_t n = -n\partial_\alpha u_\alpha \tag{102}$$

$$D_t(mu_\alpha) = -\frac{1}{n}\partial_\beta P_{\alpha\beta} - \partial_\alpha U \tag{103}$$

$$D_t\epsilon = -\frac{1}{n}\left[\partial_\alpha h_\alpha + P_{\alpha\beta}u_{\alpha\beta}\right] , \tag{104}$$

where we recall the definition of the hydrodynamic fields:

$$n = \langle 1\rangle , \qquad u_\alpha = \left\langle\frac{p_\alpha}{m}\right\rangle , \qquad \epsilon = \left\langle\frac{m}{2}\left(\frac{p_\beta}{m} - u_\beta\right)\left(\frac{p_\beta}{m} - u_\beta\right)\right\rangle , \tag{105}$$

$$P_{\alpha\beta} = mn\left\langle\left(\frac{p_\alpha}{m} - u_\alpha\right)\left(\frac{p_\beta}{m} - u_\beta\right)\right\rangle , \qquad h_\alpha = \frac{nm}{2}\left\langle\left(\frac{p_\alpha}{m} - u_\alpha\right)\left(\frac{p_\beta}{m} - u_\beta\right)\left(\frac{p_\beta}{m} - u_\beta\right)\right\rangle \tag{106}$$

with averages computed using the formula in Eq. (78), and the material derivative

$$D_t A = \partial_t A + u_\beta\partial_\beta A . \tag{107}$$

5.2 Zeroth-order hydrodynamics

Recall that, using the local equilibrium assumption (which hinged on the separation of collision timescale from the other timescales), we found an approximate solution to the Boltzmann equation, given in Eq. (65). Let's now assume that $f(\mathbf{q}, \mathbf{p}, t)$

takes on this solution, and find out the resulting behavior of the hydrodynamic fields. With proper normalization, this "zeroeth-order solution" (denoted by superscripts $^{(0)}$) is written as

$$f^{(0)}(\mathbf{q},\mathbf{p},t) = \frac{n(\mathbf{q},t)}{\left[2\pi m k_B T(\mathbf{q},t)\right]^{3/2}} \exp\left[-\frac{|\mathbf{p}-m\mathbf{u}(\mathbf{q},t)|^2}{2mk_BT(\mathbf{q},t)}\right], \tag{108}$$

where indeed $\int d^3\mathbf{p} f^{(0)} = n$ and $\langle \mathbf{p}/m\rangle^{(0)} = u$ as we have written them.

This is a Gaussian distribution over $\mathbf{p}$ with mean $\mu_\alpha(\mathbf{q},t) \equiv m u_\alpha(\mathbf{q},t)$ and variance $\sigma(\mathbf{q},t)^2 \equiv mk_BT(\mathbf{q},t)$. This makes calculation of various correlations between the components p_α relatively easy. For instance,

$$P^{(0)}_{\alpha\beta}(\mathbf{q},t) = mn\left\langle\left(\frac{p_\alpha}{m}-u_\alpha\right)\left(\frac{p_\beta}{m}-u_\beta\right)\right\rangle^{(0)} = \frac{mn}{(2\pi\sigma^2)^{3/2}}\int_{\mathbf{p}}\frac{1}{m^2}(p_\alpha-\mu_\alpha)(p_\beta-\mu_\beta)e^{-|\mathbf{p}-\boldsymbol{\mu}|^2/2\sigma^2} \tag{109}$$

$$= \frac{n}{m}\int_{\mathbf{x}} x_\alpha x_\beta \frac{e^{-|\mathbf{x}|^2/2\sigma^2}}{(2\pi\sigma^2)^{3/2}} = \frac{n^2}{m}\delta_{\alpha\beta}\sigma^2 = \delta_{\alpha\beta} n k_B T\ , \tag{110}$$

which is the ideal gas law. Likewise, we can note that

$$\epsilon(\mathbf{q},t) = \frac{1}{n}\mathrm{tr}\ P_{\alpha\beta}(\mathbf{q},t) \quad\Longrightarrow\quad \epsilon^{(0)}(\mathbf{q},t) = \frac{3}{2}k_BT \tag{111}$$

and moreover

$$h^{(0)}_\alpha = 0 \tag{112}$$

because odd moments of $p_\alpha - mu_\alpha$ are zero. Thus, we find zeroth-order hydrodynamic equations

$$D_t n = -n\partial_\alpha u_\alpha \tag{113}$$

$$mD_t u_\alpha = -\partial_\alpha U - \frac{1}{n}\partial_\alpha(nk_BT) \tag{114}$$

$$D_t\epsilon = -k_BT\partial_\alpha u_\alpha\ . \tag{115}$$

However, this description has a few deficiencies. First, it describes isentropic dynamics (as you showed in the pset), and thus can't describe a system that increases in entropy towards equilibrium. Second, none of the conserved quantities will relax to equilibrium. (I skip this calculation, which is given in lecture 11 of Mehran Kardar's ocw notes.)

5.3 First-order hydrodynamics

It is clear that we have to go beyond zeroth order. Recall our perturbative calculation, which we started in Eqs. (71)-(74). To first order in $\varepsilon = \tau_{\mathrm{MF}}/\tau_U$, we must now solve

$$\mathcal{C}[f^{(0)}, f^{(1)}] + \mathcal{C}[f^{(1)}, f^{(0)}] + \mathcal{L}[f^{(0)}]\ . \tag{116}$$

To make this calculation, we will define

$$c_\alpha \equiv \frac{p_\alpha}{m} - u_\alpha \tag{117}$$

so that

$$f^{(0)}(\mathbf{q},t) = \frac{n}{[2mk_BT]^{3/2}}\exp\left(-\frac{mc_\alpha c_\alpha}{2k_BT}\right) \tag{118}$$

and

$$\mathcal{L} = D_t + c_\alpha\partial_\alpha - \frac{1}{m}\partial_\alpha U\frac{\partial}{\partial c_\alpha}\ . \tag{119}$$

Note that the material derivative obeys the product rule

$$D_t[AB] = AD_tB + BD_tA\ . \tag{120}$$

Because $\mathcal{L}$ is $\mathcal{O}(\varepsilon)$ relative to $\mathcal{C}$, we can assume all the hydrodynamic fields in $f^{(0)}$ obey their 0th-order hydrodynamic equations (113)-(115). Then, we find

$$D_t f^{(0)} = \frac{D_t n}{n} f^{(0)} - \frac{3}{2}\frac{D_t T}{T} f^{(0)} + \frac{m(D_t u_\alpha)c_\alpha}{k_B T} f^{(0)} + \frac{c_\alpha c_\alpha D_t T}{2k_B T^2} f^{(0)} \tag{121}$$

$$\implies \frac{D_t f^{(0)}}{f^{(0)}} = -\partial_\alpha u_\alpha + \partial_\alpha u_\alpha - \frac{c_\alpha}{k_B T}\left[\partial_\alpha U + \frac{1}{n}\partial_\alpha(n k_B T)\right] - \frac{m c_\alpha c_\alpha \partial_\beta u_\beta}{3 k_B T} \tag{122}$$

$$= -c_\alpha\left[\frac{\partial_\alpha U}{k_B T} + \frac{\partial_\alpha n}{n} + \frac{\partial_\alpha T}{T} + \frac{m c_\alpha \partial_\beta c_\beta}{3 k_B T}\right] . \tag{123}$$

Similarly, $\partial_\alpha f^{(0)}$ is given by

$$\frac{\partial_\alpha f^{(0)}}{f^{(0)}} = \frac{\partial_\alpha n}{n} - \frac{3}{2}\frac{\partial_\alpha T}{T} + \frac{m(\partial_\alpha c_\beta)c_\beta}{k_B T} + \frac{m c_\beta c_\beta \partial_\alpha T}{2 k_B T^2} \tag{124}$$

and finally

$$\frac{1}{f^{(0)}}\frac{\partial}{\partial c_\alpha} f^{(0)} = -\frac{m c_\alpha}{k_B T} . \tag{125}$$

Combining these, we find

$$\frac{\mathcal{L}[f^{(0)}]}{f^{(0)}} = \left(\frac{mc^2}{2k_B T} - \frac{5}{2}\right)\frac{c_\alpha \partial_\alpha T}{T} - \frac{mc^2 \partial_\alpha c_\alpha}{3 k_B T} + \frac{m c_\alpha c_\beta \partial_\alpha c_\beta}{k_B T} \tag{126}$$

$$= \left(\frac{mc^2}{2k_B T} - \frac{5}{2}\right)\frac{c_\alpha \partial_\alpha T}{T} + \frac{m}{k_B T}\left(c_\alpha c_\beta \partial_\beta - \frac{1}{3}c^2 \partial_\alpha\right) c_\alpha \tag{127}$$

$$= \left(\frac{mc^2}{2k_B T} - \frac{5}{2}\right)\frac{c_\alpha \partial_\alpha T}{T} + \frac{m}{k_B T}\left(c_\alpha c_\beta - \frac{1}{3}c^2 \delta_{\alpha\beta}\right) u_{\alpha\beta} . \tag{128}$$

Thus, Eq. (116) becomes

$$0 = \mathcal{C}[f^{(0)}, f^{(1)}] + \mathcal{C}[f^{(1)}, f^{(0)}] + f^{(0)}\left[\left(\frac{mc^2}{2k_B T} - \frac{5}{2}\right)\frac{c_\alpha \partial_\alpha T}{T} + \frac{m}{k_B T}\left(c_\alpha c_\beta - \frac{1}{3}c^2 \delta_{\alpha\beta}\right) u_{\alpha\beta}\right] . \tag{129}$$

Let's also recall the expression for the collision integrals (where we omit all $\mathbf{q}$-dependence):

$$\mathcal{C}[f^{(0)}, f^{(1)}] + \mathcal{C}[f^{(1)}, f^{(0)}] = \int d^3\tilde{\mathbf{p}} d^2\mathbf{q}_\perp \left[\underbrace{f^{(0)}(\mathbf{p}) f^{(1)}(\tilde{\mathbf{p}})}_{\equiv 1} - \underbrace{f^{(0)}(\mathbf{p}') f^{(1)}(\tilde{\mathbf{p}}')}_{\equiv 2} + \underbrace{f^{(1)}(\mathbf{p}) f^{(0)}(\tilde{\mathbf{p}})}_{\equiv 3} - \underbrace{f^{(1)}(\mathbf{p}') f^{(0)}(\tilde{\mathbf{p}}')}_{\equiv 4}\right] . \tag{130}$$

Physically, this describes the flux of collisions between the two fields $f^{(0)}$ and its correction $f^{(1)}$. $\mathcal{C}[f^{(0)}, f^{(1)}]$ describes how collisions between $f^{(0)}$ at momentum $\mathbf{p}$ and $f^{(1)}$ at all other momenta $\tilde{\mathbf{p}}$ transfer these momenta to $\mathbf{p}'$, $\tilde{\mathbf{p}}'$. Likewise, $\mathcal{C}[f^{(1)}, f^{(0)}]$ describes how collisions between $f^{(1)}$ at momentum $\mathbf{p}$ with $f^{(0)}$ at all other momenta $\tilde{\mathbf{p}}$ transfer these momenta to $\mathbf{p}'$, $\tilde{\mathbf{p}}'$. Note that the third and term can be written as

$$3 = \int d^3\tilde{\mathbf{p}} d^2\mathbf{q}_\perp f^{(1)}(\mathbf{q}, \mathbf{p}, t) f^{(0)}(\mathbf{q}, \tilde{\mathbf{p}}, t) = f^{(1)}(\mathbf{q}, \mathbf{p}, t) . \tag{131}$$

Note also that we can enforce the normalization condition

$$n(\mathbf{q}, t) = \int d^3\mathbf{p} f(\mathbf{q}, \mathbf{p}, t) = \int d^3\mathbf{p}\left[f^{(0)}(\mathbf{q}, \mathbf{p}, t) + \varepsilon f^{(1)}(\mathbf{q}, \mathbf{p}, t) + \mathcal{O}(\varepsilon^2)\right] = n(\mathbf{q}, \mathbf{p}, t) + \varepsilon \int d^3\mathbf{p} f^{(1)}(\mathbf{q}, \mathbf{p}, t) + \mathcal{O}(\varepsilon^2) \tag{132}$$

$$\implies 0 = \int d^3\mathbf{p} f^{(1)}(\mathbf{q}, \mathbf{p}, t) \tag{133}$$

to find that the first term is zero:

$$1 = \int d^3\tilde{\mathbf{p}} d^2\mathbf{q}_\perp f^{(0)}(\mathbf{p}) f^{(1)}(\tilde{\mathbf{p}}) = f^{(0)}(\mathbf{p}) \int d^3\tilde{\mathbf{p}} d^2\mathbf{q}_\perp f^{(1)}(\tilde{\mathbf{p}}) = 0 . \tag{134}$$

Through a more hand-wavey argument, we can also rationalize that because the 2nd and 4th term also integrate over the argument of $f^{(1)}$, they should also be zero

$$② , ④ \approx 0\ . \tag{135}$$

(This is more true when, for instance, the interactions are weaker.) Thus, we can approximate

$$\mathcal{C}[f^{(0)}, f^{(1)}] + \mathcal{C}[f^{(1)}, f^{(0)}] \approx ③ = f^{(1)}(\mathbf{q}, \mathbf{p}, t) \tag{136}$$

$$\implies 0 = f^{(1)}(\mathbf{q}, \mathbf{p}, t) + f^{(0)}\left[\left(\frac{mc^2}{2k_BT} - \frac{5}{2}\right)\frac{c_\alpha \partial_\alpha T}{T} + \frac{m}{k_BT}\left(c_\alpha c_\beta - \frac{1}{3}c^2\delta_{\alpha\beta}\right)u_{\alpha\beta}\right]\ . \tag{137}$$

Thus, we find the 1st-order correction to the density

$$f^{(1)}(\mathbf{q}, \mathbf{p}, t) = -f^{(0)}(\mathbf{q}, \mathbf{p}, t)\left[\frac{m}{k_BT}\left(c_\alpha c_\beta - \frac{\delta_{\alpha\beta}}{3}c^2\right)u_{\alpha\beta} - \left(\frac{mc^2}{2k_BT} - \frac{5}{2}\right)c_\alpha\partial_\alpha \ln T\right] \tag{138}$$

so that

$$f(\mathbf{q}, \mathbf{p}, t) = f^{(0)}(\mathbf{q}, \mathbf{p}, t)\left\{1 - \varepsilon\left[\frac{m}{k_BT}\left(c_\alpha c_\beta - \frac{\delta_{\alpha\beta}}{3}c^2\right)u_{\alpha\beta} - \left(\frac{mc^2}{2k_BT} - \frac{5}{2}\right)c_\alpha\partial_\alpha \ln T\right]\right\} + \mathcal{O}(\varepsilon^2) \tag{139}$$

$$f^{(0)}(\mathbf{q}, \mathbf{p}, t) = \frac{n(\mathbf{q}, t)}{[2\pi m k_BT(\mathbf{q}, t)]^{3/2}}\exp\left(-\frac{mc_\alpha(\mathbf{q}, \mathbf{p}, t)c_\alpha(\mathbf{q}, \mathbf{p}, t)}{2k_BT(\mathbf{q}, t)}\right) \tag{140}$$

where $c_\alpha(\mathbf{q}, \mathbf{p}, t) = p_\alpha/m - u_\alpha(\mathbf{q}, t)$.

Let's clean up our notation by partially changing variables to $\mathbf{c} = \mathbf{p}/m - \mathbf{u}$ (while keeping in mind $\mathbf{c}$'s hidden $\mathbf{q}$-dependence) and changing from the temperature to the "variance" $v(\mathbf{q}, t) \equiv k_BT(\mathbf{q}, t)/m$, so that

$$f^{(0)}(\mathbf{q}, \mathbf{c}, t) = n(\mathbf{q}, t)\left[\frac{m}{2\pi k_BT(\mathbf{q}, t)}\right]^{3/2}\exp\left(-\frac{mc_\alpha(\mathbf{q}, \mathbf{p}, t)c_\alpha(\mathbf{q}, \mathbf{p}, t)}{2k_BT(\mathbf{q}, t)}\right) \equiv \frac{n(\mathbf{q}, t)}{[2\pi v(\mathbf{q}, t)]^{3/2}}\exp\left(-\frac{c^2}{2v(\mathbf{q}, t)}\right) \tag{141}$$

$$-\frac{f^{(1)}(\mathbf{q}, \mathbf{c}, t)}{f^{(0)}(\mathbf{q}, \mathbf{c}, t)} = \left(c_\alpha c_\beta - \frac{\delta_{\alpha\beta}}{3}c^2\right)\frac{u_{\alpha\beta}}{v} - \left(\frac{c^2}{2v} - \frac{5}{2}\right)c_\alpha\partial_\alpha \ln v\ . \tag{142}$$

We also define the 1st-order average

$$\langle \mathcal{O}(\mathbf{q}, \mathbf{c}, t)\rangle^{(1)} = \frac{1}{n(\mathbf{q}, t)}\int d^3\mathbf{c} f^{(1)}(\mathbf{q}, \mathbf{c}, t)\mathcal{O}(\mathbf{q}, \mathbf{c}, t) = \left\langle \mathcal{O}\left[\left(c_\alpha c_\beta - \frac{\delta_{\alpha\beta}}{3}c^2\right)\frac{u_{\alpha\beta}}{v} - \left(\frac{c^2}{2v} - \frac{5}{2}\right)c_\alpha\partial_\alpha \ln v\right]\right\rangle^{(0)} \tag{143}$$

$$\implies \langle \mathcal{O}\rangle = \langle \mathcal{O}\rangle^{(0)} + \varepsilon\langle \mathcal{O}\rangle^{(1)} + \mathcal{O}(\varepsilon^2)\ . \tag{144}$$

We can then use Wick's theorem to calculate the 1st-order correction to the momentum:

$$\left\langle \frac{p_\alpha}{m}\right\rangle^{(1)} = \left\langle (c_\alpha + u_\alpha)\left[\left(c_\gamma c_\beta - \frac{\delta_{\gamma\beta}}{3}c^2\right)\frac{u_{\gamma\beta}}{v} - \left(\frac{c^2}{2v} - \frac{5}{2}\right)c_\gamma\partial_\gamma \ln v\right]\right\rangle^{(0)} \tag{145}$$

$$= \left\langle u_\alpha\left(c_\gamma c_\beta - \frac{\delta_{\gamma\beta}}{3}c^2\right)\frac{u_{\gamma\beta}}{v} + \left(\frac{c^2}{2v} - \frac{5}{2}\right)c_\alpha c_\gamma\partial_\gamma \ln v\right\rangle^{(0)} \tag{146}$$

$$= \left(2\delta_{\beta\gamma} - \frac{\delta_{\gamma\beta}}{3}6\right)\frac{u_{\gamma\beta}}{v} + \left(\frac{1}{2v}\delta_{\alpha\gamma}(3+2)v^2 - \frac{5}{2}\cdot 2v\delta_{\alpha\gamma}\right)\partial_\gamma \ln v = 0 \tag{147}$$

and the pressure:

$$P^{(1)}_{\alpha\beta} = nm\langle c_\alpha c_\beta\rangle^{(1)} = nm\left\langle -c_\alpha c_\beta\left[\left(c_\gamma c_\lambda - \frac{\delta_{\gamma\lambda}}{3}c^2\right)\frac{u_{\gamma\lambda}}{v} - \left(\frac{c^2}{2v} - \frac{5}{2}\right)c_\gamma\partial_\gamma \ln v\right]\right\rangle^{(0)} \tag{148}$$

$$= nm\left\langle -c_\alpha c_\beta\left(c_\gamma c_\lambda - \frac{\delta_{\gamma\lambda}}{3}c^2\right)\frac{u_{\gamma\lambda}}{v}\right\rangle^{(0)} \tag{149}$$

$$= -nm\frac{u_{\gamma\lambda}}{v}\left(v^2\left(\delta_{\alpha\beta}\delta_{\gamma\lambda} + \delta_{\alpha\gamma}\delta_{\beta\lambda} + \delta_{\alpha\lambda}\delta_{\beta\gamma}\right) - \frac{\delta_{\gamma\lambda}}{3}v^2(3+2)\delta_{\alpha\beta}\right) \tag{150}$$

$$= -nmv\left[2u_{\alpha\beta} - \frac{2}{3}\delta_{\alpha\beta}u_{\gamma\gamma}\right] = -2nk_BT\left[u_{\alpha\beta} - \frac{\delta_{\alpha\beta}}{3}u_{\gamma\gamma}\right] \tag{151}$$

and the energy:

$$\epsilon^{(1)} = \frac{m}{2}\langle c_\alpha c_\alpha\rangle^{(1)} = \frac{1}{n}\mathrm{tr}\ P^{(1)}_{\alpha\beta} = 0 \tag{152}$$

and finally the heat flux:

$$h^{(1)}_\alpha = \frac{nm}{2}\langle c_\alpha c^2\rangle^{(1)} = -\frac{nm}{2}\left\langle c_\alpha c^2\left(\frac{c^2}{2v} - \frac{5}{2}\right)c_\gamma\partial_\gamma \ln v\right)\right\rangle^{(0)} \tag{153}$$

$$= -\frac{nm}{4}\partial_\gamma \ln v\left\langle \frac{c_\beta c_\beta c_\lambda c_\lambda c_\alpha c_\gamma}{v} - 5c_\beta c_\beta c_\alpha c_\gamma\right\rangle \tag{154}$$

$$= -\frac{nmv\partial_\gamma v}{4}\Big[35 - 5\cdot 5\Big]\delta_{\alpha\gamma} = -\frac{5}{2}\frac{nk_B^2 T\partial_\alpha T}{m}\ . \tag{155}$$

In summary, we see that the hydrodynamic equations (86), (93), and (100) become at first order

$$D_t n = -n\partial_\alpha u_\alpha \tag{156}$$

$$D_t(mu_\alpha) = -\frac{1}{n}\partial_\beta\Big[nk_BT\Big(\delta_{\alpha\beta} - 2\varepsilon(u_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}u_{\gamma\gamma})\Big)\Big] - \partial_\alpha U \tag{157}$$

$$D_t\epsilon = \frac{1}{n}\left[\varepsilon\frac{5}{2}\partial_\alpha\left(\frac{nk_B^2T\partial_\alpha T}{m}\right) - nk_BT\Big(\delta_{\alpha\beta} - 2\varepsilon(u_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta}u_{\gamma\gamma})\Big)u_{\alpha\beta}\right]\ . \tag{158}$$

In the interest of time, we stopped around here.